Energy and potential enstrophy flux constraints in the two-layer quasi-geostrophic model

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We investigate an inequality constraining the energy and potential enstrophy flux in the two-layer quasi-geostrophic model. This flux inequality is unconditionally satisfied for the case of two-dimensional Navier-Stokes turbulence. However, it is not obvious that it remains valid under the multi-layer quasi-geostrophic model. The physical significance of this inequality is that it decides whether any given model can reproduce the Nastrom-Gage spectrum of the atmosphere, at least in terms of the total energy spectrum. We derive the general form of the energy and potential enstrophy dissipation rate spectra for a generalized multi-layer model. We then specialize these results for the case of the two-layer quasi-geostrophic model under dissipation configurations in which the dissipation terms for each layer are dependent only on the streamfunction or potential vorticity of that layer. We derive sufficient conditions for satisfying the flux inequality and discuss the possibility of violating it under different conditions.

1. Introduction

It is now well-known that in two-dimensional Navier-Stokes turbulence, most of the energy tends to go towards larges scales and most of the enstrophy tends to go towards small scales. This was initially proposed by Fjørtøft (1953) via his triad interactions argument. Later, Kraichnan (1967), Leith (1968), and Batchelor (1969) introduced the theory that the energy forms an upscale inverse energy cascade with energy spectrum scaling as $k^{-5/3}$, and that the enstrophy forms a downscale enstrophy cascade with k^{-3} scaling. Kraichnan (1967) argued, differently from Fjørtøft (1953), that the direction of the two cascades can be justified via a thermodynamic argument in which we introduce, without proof, the assumption that the energy and enstrophy fluxes should tend to revert the energy spectrum from a cascade configuration to the absolute equilibrium configuration. The existence of forcing and dissipation arrests this tendency, thus keeping the system locked in a steady-state forced-dissipative configuration away from absolute equilibrium.

Merilees & Warn (1975) identified a serious error with the original Fjørtøft argument: Fjørtøft (1953) claimed that in every triad interaction group, more energy is transferred upscale than downscale. However, a more rigorous analysis shows that there exist triad interaction groups in which more energy is sent downscale than upscale, and it is not obvious, without additional considerations, as to which group is dominant. This was explained in detail also by Gkioulekas & Tung (2006). Aside from this matter, the fundamental problem that underlies every other proof that utilizes only the twin conservation laws of enstrophy and energy, is that an additional assumption needs to be introduced to

overcome the symmetry of the Euler equations under time reversal. Typical assumptions, such as the tendency of the energy spectrum to revert to absolute equilibrium, or the tendency of an energy peak to spread, typify ad hoc constraints imposed implicitly on the initial conditions that are needed to break the time reversal symmetry. All of these proofs have been reviewed in detail by Gkioulekas & Tung (2007*b*). More importantly, Gkioulekas & Tung (2007*b*) have counterproposed a very simple and mathematically rigorous proof that avoids the need for any ad hoc assumptions by considering the combined effect of the Navier-Stokes nonlinearity and the dissipation terms. The only assumption used by this proof is that the forcing spectrum is restricted to a finite interval $[k_1, k_2]$ of wavenumbers. Later, Farazmand (2010) and Farazmand, Kevlahan & Protas (2011) showed that even that assumption can be relaxed to some extent, although not entirely eliminated.

The essence of the argument is to show that for every wavenumber k not in the forcing range, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ satisfy the inequality $k^2\Pi_E(k) - \Pi_G(k) \leq 0$. Here, $\Pi_E(k)$ represents the amount of energy per unit volume transferred from the wavenumbers in the $(0, k)$ interval to the wavenumbers in the $(k, +\infty)$ interval, and $\Pi_G(k)$ is defined similarly for the enstrophy. From this inequality we then derive the following integral constraints for $\Pi_E(k)$ and $\Pi_G(k)$:

$$\int_0^k q\Pi_E(q)\,\mathrm{d}q \leq 0,\ \forall k \in (k_2, +\infty), \tag{1.1}$$

$$\int_k^{+\infty} q^{-3}\Pi_G(q)\,\mathrm{d}q \geq 0,\ \forall k \in (0, k_1). \tag{1.2}$$

As explained by Gkioulekas & Tung (2007*b*), these constraints imply a predominantly upscale transfer of energy and a predominantly downscale transfer of enstrophy. The original flux inequality $k^2\Pi_E(k) - \Pi_G(k) < 0$ itself can also be directly interpreted as a tight constraint on the downscale energy flux.

The question that concerns us in the present paper is whether this flux inequality continues to hold for multi-layer quasi-geostrophic models. Quasi-geostrophic turbulence, as shown recently by Lindborg (2007), is a reasonable approximation for the cascade dynamics of atmospheric turbulence at length-scales larger than 100km, in order of magnitude. Underlying the model are the assumptions of small thickness and fast rotation. Towards large scales, the atmospheric thickness becomes relatively small and the Coriolis effect of rotation becomes increasingly noticeable. In multi-layer quasi-geostrophic models, the vertical dimension is discretized into a finite number of layers. The simplest layer model is one in which we only have two layers. This two-layer quasi-geostrophic model was simulated by Tung & Orlando (2003), who claimed that it can reproduce the energy spectrum of the atmosphere, consistently with the measurement of Gage & Nastrom (1986). Tung & Orlando (2003) intended to corroborate their theory that the Nastrom-Gage spectrum consists of a downscale potential enstrophy cascade coexisting with a downscale energy cascade over the same range of scales. Gkioulekas & Tung (2005*a*,*b*) clarified the theoretical mechanism that allows cascades to coexist and predicted that each cascade provides a contributing term to the energy spectrum and that both contributions are linearly superposed with one another. This linear superposition hypothesis and the Tung-Orlando theory of coexisting cascades have been both corroborated by the recent measurements and analysis by Terasaki, Tanaka & Zagar (2011). A recent primitive equations numerical simulation by Vallgren, Deusebio & Lindborg (2011) also seems to support the notion of a downscale energy cascade coexisting with a downscale potential enstrophy cascade, and a more detailed discussion of these matters has been given by Gkioulekas (2012).

More specifically, the Nastrom Gage spectrum of the atmosphere consists of a k^{-3} slope in the energy spectrum that transitions to $k^{-5/3}$ at a length scale of approximately 1000km to 700km, near the Rossby deformation wavenumber k_R. According to the theory of Tung & Orlando (2003) and Gkioulekas & Tung (2005*a*,*b*), the Nastrom-Gage spectrum arises from coexisting downscale cascades of potential enstrophy and energy, over the same range of scales, each cascade gives a contribution to the energy spectrum, and both contributions are combined linearly. As a result, the k^{-3} contribution of the downscale potential enstrophy cascade is dominant for small wave numbers k, only to be overtaken by the $k^{-5/3}$ contribution of the downscale energy cascade near a transition wavenumber k_t. For the $k^{-5/3}$ contribution of the downscale energy cascade to become dominant at wavenumbers $k \sim k_t$ beyond a transition wavenumber k_t, it is necessary for the flux inequality to break down near k_t and reverse direction for wavenumbers $k > k_t$. *Consequently, the direction of the flux inequality decides whether various models can reproduce the Nastrom-Gage spectrum, at least in terms of the total energy.*

For the case of two-dimensional turbulence, as was first noted by Gkioulekas & Tung (2005*a*,*b*), the unconditional validity of the flux inequality constitutes a mathematically rigorous proof that two-dimensional turbulence cannot reproduce the Nastrom-Gage spectrum under any circumstances, as long as the dissipation terms are given in terms of integer or fractional powers of Laplacian operators. For the case of the real atmosphere, it is fairly obvious that the flux inequality is violated, provided that we agree with the Tung-Orlando theory (Tung & Orlando 2003) that the Nastrom-Gage spectrum consists of two coexisting downscale cascades of potential enstrophy and energy. If the flux inequality can be also violated by quasi-geostrophic models, then these models can reproduce the Nastrom-Gage energy spectrum, at least in terms of the total energy spectrum $E(k)$, if not in terms of a correct distribution of energy between kinetic energy and potential energy. If, on the other hand, the flux inequality continues to hold, then quasi-geostrophic models cannot reproduce the Nastrom-Gage spectrum either. The question that we ultimately want to investigate is whether the quasi-geostrophic models *have enough physics* to enable a break-down of the flux inequality, and what configuration of the dissipation terms is needed to accomplish that.

Gkioulekas & Tung (2007*a*) first noticed that the only mechanism that can cause a flux inequality violation is an asymmetry in the dissipation operators for different layers. Unfortunately, without more information about the phenomenology of the two-layer quasi-geostrophic model, and multi-layer quasi-geostrophic models in general, it is very difficult to derive any sufficient conditions for the needed flux inequality breakdown. On the other hand, it is possible to derive rigorous sufficient conditions for satisfying the flux inequality when the asymmetry is restricted by an upper bound, without assuming any knowledge whatsoever of the phenomenology of any of the spectra involved. The goal of the present paper is to address this non-controversial aspect of the larger problem, by deriving a series of such results, for the case of weak dissipation asymmetry, and in doing so, to also set up the necessary mathematical infrastructure for a further exploration of the flux inequality. Unlike the case of two-dimensional turbulence, where the derivation of the flux inequality is easy, it should be clear from the present manuscript that the flux inequality in the quasi-geostrophic model is both nontrivial and sensitive to the configuration of the dissipation terms across layers.

This paper is organized as follows. In section 2, we introduce the generalized multi-layer model and its conservation laws of energy and layer-by-layer potential enstrophy. In section 3, we introduce the budget equations for the energy and potential enstrophy spectra, derive the required conditions for the energy spectrum definition to remain positive definite, and derive general expressions for the energy and potential enstrophy

dissipation rate spectra. Simplifications of these general expressions are given for the special cases of streamfunction dissipation and potential vorticity dissipation. From these results, in section 4 we derive sufficient conditions for satisfying the flux inequality for the special case of the two-layer quasi-geostrophic model under the two special cases of streamfunction dissipation and potential vorticity dissipation. Because the details of the calculation are tedious, the main results are summarized in section 5. Conclusions and general discussion are given in section 6. Technical matters are taken up in the appendices.

2. The generalized multilayer model and conservation laws

Following Gkioulekas (2012), we write the governing equations for the generalized multi-layer model in matrix form:

$$\frac{\partial q_\alpha}{\partial t} + J(\psi_\alpha, q_\alpha) = d_\alpha + f_\alpha, \tag{2.1}$$

$$d_\alpha = \sum_\beta \mathscr{D}_{\alpha\beta}\psi_\beta. \tag{2.2}$$

Here ψ_α represents the streamfunction at the α-layer, q_α represents the potential vorticity at the α-layer, $\mathscr{D}_{\alpha\beta}$ is a linear operator encapsulating the dissipation terms, and f_α is the forcing term acting on the α-layer. The index α takes the values $\alpha = 1, 2, \ldots, n$ representing the layer number, for a model involving n layers. Sums over indices, such as in the sum over the index β in the dissipation terms above, are assumed to run over all layers $1, 2, \ldots, n$, unless we indicate otherwise. It is also assumed that the streamfunction ψ_α and the potential vorticity q_α are related via a linear operator $\mathscr{L}_{\alpha\beta}$ according to:

$$q_\alpha(\mathbf{x}, t) = \sum_\beta \mathscr{L}_{\alpha\beta}\psi_\alpha(\mathbf{x}, t). \tag{2.3}$$

The above equations encompass both the two-layer quasi-geostrophic model and the multilayer quasi-geostrophic model, on the assumption that we neglect the β-effect, arising from the latitudinal dependence of the Coriolis pseudoforce. This is a reasonable assumption for Earth, especially if we restrict our interest to a thin strip of the Earth's surface, oriented parallel to the equator. Baroclinicity instability is accounted for by the forcing term f_α, and implicit in the entire argument is the assumption that it forces the system at large scales only. This assumption, originally proposed by Salmon (1978, 1980), is the only physical assumption implicit in the theoretical framework of the flux inequality, and it has been corroborated by Welch & Tung (1998) and Tung & Orlando (2003).

For the sake of simplifying our analysis, we assume that all fields are defined in an infinite two-dimensional domain. Then we can write the Fourier expansions for the streamfunction ψ and the potential vorticity q as follows:

$$\psi_\alpha(\mathbf{x}, t) = \int_{\mathbb{R}^2} \hat{\psi}_\alpha(\mathbf{k}, t)\exp(i\mathbf{k}\cdot\mathbf{x})\,\mathrm{d}\mathbf{k}, \tag{2.4}$$

$$q_\alpha(\mathbf{x}, t) = \int_{\mathbb{R}^2} \hat{q}_\alpha(\mathbf{k}, t)\exp(i\mathbf{k}\cdot\mathbf{x})\,\mathrm{d}\mathbf{k}. \tag{2.5}$$

We assume that the operator $\mathscr{L}_{\alpha\beta}$ is diagonal in Fourier space. This means that the relation between the streamfunction and the potential vorticity, in Fourier space, reads:

$$\hat{q}_\alpha(\mathbf{k}, t) = \sum_\beta L_{\alpha\beta}(\|\mathbf{k}\|)\hat{\psi}_\alpha(\mathbf{k}, t). \tag{2.6}$$

We also assume that $\mathscr{L}_{\alpha\beta}$ is symmetric with $\mathscr{L}_{\alpha\beta} = \mathscr{L}_{\beta\alpha}$. This implies that $L_{\alpha\beta}(k) = L_{\beta\alpha}(k)$ for all wavenumbers k. For an n-layer quasi-geostrophic model, the $L_{\alpha\beta}(k)$ matrix becomes a tridiagonal matrix. For the special case of the two-layer quasi-geostrophic model, the matrix $L_{\alpha\beta}(k)$ is given by

$$L_{\alpha\beta}(k) = \begin{bmatrix} -k^2 - k_R^2/2 & +k_R^2/2 \\ +k_R^2/2 & -k^2 - k_R^2/2 \end{bmatrix}, \tag{2.7}$$

with k_R the Rossby wavenumber. For quasi-geostrophic models, the matrix $L_{\alpha\beta}(k)$ is non-singular for all wavenumbers $k > 0$, due to being diagonally dominant, and we assume that to be the case in our abstract formulation given above. Consequently, there is an inverse matrix $L_{\alpha\beta}^{-1}(k)$ which defines the inverse operator $\mathscr{L}_{\alpha\beta}^{-1}$. To accommodate a possible singularity at $k = 0$ we assume that at wavenumber $k = 0$, in Fourier space, the corresponding field component is 0 for all fields. This is equivalent to subtracting the mean field and considering only the field fluctuation around the mean.

Likewise, we assume that the dissipation operation $\mathscr{D}_{\alpha\beta}$ is also diagonal in Fourier space and that the Fourier expansion of the dissipation term $\mathscr{D}_{\alpha\beta}\psi_\beta$ reads:

$$(\mathscr{D}_{\alpha\beta}\psi_\beta)(\mathbf{x}, t) = \int_{\mathbb{R}^2} D_{\alpha\beta}(\|\mathbf{k}\|)\hat{\psi}_\beta(\mathbf{k}, t) \exp(i\mathbf{k}\cdot\mathbf{x})\, \mathrm{d}\mathbf{k}. \tag{2.8}$$

In general, the dissipation operator is a matrix operator that is allowed to entangle multiple layers together. Realistically, this occurs when one follows Salmon (1978, 1980) in the definition of the Ekman term, which should appear only on the bottom layer but entangles both layers together. Previously, Tung & Orlando (2003), Gkioulekas & Tung (2007*a*), and Gkioulekas (2012), restricted their attention to diagonal dissipation only where $D_{\alpha\beta}(k) = 0$ for $\alpha \neq \beta$. In this case, the spectrum $D_{\alpha\beta}(k)$ of the dissipation operator reads $D_{\alpha\beta}(k) = \delta_{\alpha\beta} D_\beta(k)$ with $\delta_{\alpha\beta}$ given by

$$\delta_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta \end{cases}. \tag{2.9}$$

We will now show that the generalized layer model, in the absence of dissipation, conserves the total energy E and the total potential enstrophy G under very general conditions on the operator $\mathscr{L}_{\alpha\beta}$, For any arbitrary scalar field $f(x, y)$ we write the corresponding volume integral using the following notation:

$$\langle\langle f \rangle\rangle = \iint_{\mathbb{R}^2} f(x, y)\, \mathrm{d}x\mathrm{d}y. \tag{2.10}$$

We define the total energy E over all layers, and the layer-by-layer total potential enstrophy G_α for layer α, as $E = -\sum_\alpha \langle\langle \psi_\alpha q_\alpha \rangle\rangle$ and $G_\alpha = \langle\langle q_\alpha^2 \rangle\rangle$. The purpose of the minus sign in our definition of the total energy E is to maintain consistency with the notation and sign conventions used by Gkioulekas (2012). Specifically, we will show that the potential enstrophy is conserved on a layer-by-layer basis unconditionally regardless of the details of the operator $\mathscr{L}_{\alpha\beta}$. Conservation of the total energy E, over all layers, on the other hand, requires that the operator $\mathscr{L}_{\alpha\beta}$ be *symmetric* and *self-adjoint*. By symmetric we mean that the operator satisfies $\mathscr{L}_{\alpha\beta} = \mathscr{L}_{\beta\alpha}$. To define the self-adjoint property, consider two arbitrary two-dimensional scalar fields $f(x, y)$ and $g(x, y)$. We require that every component of the operator $\mathscr{L}_{\alpha\beta}$ must satisfy $\langle\langle f(\mathscr{L}_{\alpha\beta} g) \rangle\rangle = \langle\langle (\mathscr{L}_{\alpha\beta} f) g \rangle\rangle$ for any two fields $f(x, y)$ and $g(x, y)$. This self-adjoint property, so defined, follows as an immediate consequence of our previous assumption that the operator $\mathscr{L}_{\alpha\beta}$ is diagonal in Fourier space. In the proof given below, however, there is no need to use the stronger assumption of diagonality.

The proof is based on the following properties of the nonlinear Jacobian term. If $a(x, y)$ and $b(x, y)$ are two-dimensional scalar fields that satisfy a homogeneous (Dirichlet or Neumann) boundary condition, then we can show that $\langle\langle J(a, b)\rangle\rangle = 0$, using integration by parts. Then, we note that, as an immediate consequence of the product rule of differentiation, given three two-dimensional scalar fields $a(x, y), b(x, y)$, and $c(x, y)$ we have

$$\langle\langle J(ab, c)\rangle\rangle = \langle\langle aJ(b, c)\rangle\rangle + \langle\langle bJ(a, c)\rangle\rangle = 0, \tag{2.11}$$

from which we obtain the identity

$$\langle\langle aJ(b, c)\rangle\rangle = \langle\langle bJ(c, a)\rangle\rangle = \langle\langle cJ(a, b)\rangle\rangle. \tag{2.12}$$

Now, let us go ahead and drop the dissipation and forcing terms and write the time-derivative of the potential vorticity q_α as $\dot{q}_\alpha = -J(\psi_\alpha, q_\alpha)$. Then, the time derivative of the streamfunction ψ_α reads:

$$\dot{\psi}_\alpha = \sum_\beta \mathscr{L}^{-1}_{\alpha\beta}\dot{q}_\beta = -\sum_\beta \mathscr{L}^{-1}_{\alpha\beta} J(\psi_\beta, q_\beta). \tag{2.13}$$

Differentiating the total potential enstrophy G_α for the α layer with respect to time and employing the identity given by Eq. (2.12) immediately gives:

$$\dot{G}_\alpha = 2\langle\langle q_\alpha \dot{q}_\alpha\rangle\rangle = -2\langle\langle q_\alpha J(\psi_\alpha, q_\alpha)\rangle\rangle = -2\langle\langle \psi_\alpha J(q_\alpha, q_\alpha)\rangle\rangle = 0. \tag{2.14}$$

Here, we note that from the definition of the Jacobian $J(q_\alpha, q_\alpha) = 0$. This establishes the layer-by-layer conservation law of potential enstrophy, unconditionally, as claimed. To show the energy conservation law, we differentiate the total energy E with respect to time and obtain:

$$\dot{E} = -(\mathrm{d}/\mathrm{d}t)\sum_\alpha \langle\langle \psi_\alpha q_\alpha\rangle\rangle = -\sum_\alpha \langle\langle \dot{\psi}_\alpha q_\alpha\rangle\rangle - \sum_\alpha \langle\langle \psi_\alpha \dot{q}_\alpha\rangle\rangle \tag{2.15}$$

$$= \sum_{\alpha\beta} \langle\langle q_\alpha \mathscr{L}^{-1}_{\alpha\beta} J(\psi_\beta, q_\beta)\rangle\rangle + \sum_\alpha \langle\langle \psi_\alpha J(\psi_\alpha, q_\alpha)\rangle\rangle \tag{2.16}$$

$$= \sum_{\alpha\beta} \langle\langle J(\psi_\beta, q_\beta) \mathscr{L}^{-1}_{\alpha\beta} q_\alpha\rangle\rangle + \sum_\alpha \langle\langle q_\alpha J(\psi_\alpha, \psi_\alpha)\rangle\rangle \tag{2.17}$$

$$= \sum_{\alpha\beta} \langle\langle J(\psi_\beta, q_\beta) \mathscr{L}^{-1}_{\beta\alpha} q_\alpha\rangle\rangle = \sum_\beta \langle\langle J(\psi_\beta, q_\beta)\psi_\beta\rangle\rangle \tag{2.18}$$

$$= \sum_\beta \langle\langle J(\psi_\beta, \psi_\beta) q_\beta\rangle\rangle = 0. \tag{2.19}$$

Note that the self-adjoint property is applied at step Eq. (2.17), and the symmetric property is applied at Eq. (2.18). This concludes the proof.

3. Spectra and budget equations

Following Frisch (1995) and Gkioulekas (2012), we define spectra for the energy and potential enstrophy using the bracket notation, which is defined as follows. Consider, in general, two arbitrary two-dimensional scalar fields $a(x)$ and $b(x)$. Let $a^{<k}(\mathbf{x})$ and $b^{<k}(\mathbf{x})$ be the fields obtained from $a(\mathbf{x})$ and $b(\mathbf{x})$ by setting to zero, in Fourier space, the components corresponding to wavenumbers whose norm is greater than k. Formally, $a^{<k}(\mathbf{x})$ is defined as

$$a^{<k}(\mathbf{x}) = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}_0 \int_{\mathbb{R}^2} \mathrm{d}\mathbf{k}_0 \, \frac{H(k - \|\mathbf{k}_0\|)}{4\pi^2} \exp(i\mathbf{k}_0 \cdot (\mathbf{x} - \mathbf{x}_0)) a(\mathbf{x}_0), \tag{3.1}$$

with $H(x)$ the Heaviside function, defined as the integral of a delta function:

$$H(x) = \int_0^x \delta(\tau)\, d\tau = \begin{cases} 1, & \text{if if } x \in (0, +\infty) \\ 1/2, & \text{if if } x = 0 \\ 0, & \text{if if } x \in (-\infty, 0) \end{cases} . \tag{3.2}$$

Obviously, $b^{<k}(\mathbf{x})$ is defined similarly. We now use the two filtered fields $a^{<k}(\mathbf{x})$ and $b^{<k}(\mathbf{x})$ to define the bracket $\langle a, b \rangle_k$ as:

$$\langle a, b \rangle_k = \frac{\mathrm{d}}{\mathrm{d}k} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\ \left\langle a^{<k}(\mathbf{x}) b^{<k}(\mathbf{x}) \right\rangle \tag{3.3}$$

$$= \frac{1}{2} \int_{A \in \mathrm{SO}(2)} \mathrm{d}\Omega(A)\ \left\langle [\hat{a}^*(kA\mathbf{e})\hat{b}(kA\mathbf{e}) + \hat{a}(kA\mathbf{e})\hat{b}^*(kA\mathbf{e})] \right\rangle . \tag{3.4}$$

Here, $\hat{a}(\mathbf{k})$ and $\hat{b}(\mathbf{k})$ are the Fourier transforms of $a(\mathbf{x})$ and $b(\mathbf{x})$, SO(2) is the set of all non-reflecting rotation matrices in two dimensions, $d\Omega(A)$ is the measure of a spherical integral, $\mathbf{e}$ is a two-dimensional unit vector, and $\langle \cdot \rangle$ represents taking an ensemble average. The star superscript represents taking the complex conjugate. Note that Eq. (3.3) is the definition of the bracket, and Eq. (3.4) follows from Eq. (3.3) as a consequence. As noted by Gkioulekas (2012), the bracket satisfies the following properties:

$$\langle a, b \rangle_k = \langle b, a \rangle_k , \tag{3.5}$$

$$\langle a, b + c \rangle_k = \langle a, b \rangle_k + \langle a, c \rangle_k , \tag{3.6}$$

$$\langle a + b, c \rangle_k = \langle a, c \rangle_k + \langle b, c \rangle_k . \tag{3.7}$$

Moreover, every $(\alpha\beta)$-component of the operator $\mathscr{L}_{\alpha\beta}$ is self-adjoint with respect to the bracket, which gives

$$\langle \mathscr{L}_{\alpha\beta} a, b \rangle_k = \langle a, \mathscr{L}_{\alpha\beta} b \rangle_k = L_{\alpha\beta}(k) \langle a, b \rangle_k , \tag{3.8}$$

and the same property is also satisfied by every component of the inverse operator $\mathscr{L}^{-1}_{\alpha\beta}$:

$$\left\langle \mathscr{L}^{-1}_{\alpha\beta} a, b \right\rangle_k = \left\langle a, \mathscr{L}^{-1}_{\alpha\beta} b \right\rangle_k = L^{-1}_{\alpha\beta}(k) \langle a, b \rangle_k . \tag{3.9}$$

Using the bracket, we define the energy spectrum $E(k) = -\sum_\alpha \langle \psi_\alpha, q_\alpha \rangle_k$, and we also define the layer-by-layer potential enstrophy spectrum $G_\alpha(k) = \langle q_\alpha, q_\alpha \rangle_k$ and the total potential enstrophy spectrum $G(k) = \sum_\alpha G_\alpha(k)$. Unlike the case of two-dimensional Navier-Stokes, where the enstrophy and energy spectra $G(k)$ and $E(k)$ are related via a simple equation, $G(k) = k^2 E(k)$, in the generalized layer model, the potential enstrophy spectrum and the energy spectrum are related indirectly, as shown below:

Define the streamfunction spectrum $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$. Then, via the properties of the bracket above, the energy spectrum $E(k)$ reads

$$E(k) = -\sum_\alpha \langle \psi_\alpha, q_\alpha \rangle_k = -\sum_\alpha \left\langle \psi_\alpha, \sum_\beta \mathscr{L}_{\alpha\beta} \psi_\beta \right\rangle_k = -\sum_{\alpha\beta} L_{\alpha\beta}(k) \langle \psi_\alpha, \psi_\beta \rangle_k \tag{3.10}$$

$$= -\sum_{\alpha\beta} L_{\alpha\beta}(k) C_{\alpha\beta}(k), \tag{3.11}$$

and the potential enstrophy spectrum $G_\alpha(k)$ reads

$$G(k) = \sum_\alpha \langle q_\alpha, q_\alpha \rangle_k = \sum_\alpha \left\langle \sum_\beta \mathscr{L}_{\alpha\beta} \psi_\beta, \sum_\gamma \mathscr{L}_{\alpha\gamma} \psi_\gamma \right\rangle_k \tag{3.12}$$

$$= \sum_{\alpha\beta} L_{\alpha\beta}(k) \left\langle \psi_\beta, \sum_\gamma \mathscr{L}_{\alpha\gamma}\psi_\gamma \right\rangle_k = \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) \left\langle \psi_\beta, \psi_\gamma \right\rangle_k \tag{3.13}$$

$$= \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) C_{\beta\gamma}(k). \tag{3.14}$$

Thus, they are related only indirectly via the streamfunction spectrum $C_{\alpha\beta}(k)$. This was noted previously by Gkioulekas (2012).

We note that for $\alpha \neq \beta$, $C_{\alpha\beta}(k)$ is not positive-definite and may take positive or negative values. For the case $\alpha = \beta$ we define $U_\alpha(k) = \langle \psi_\alpha, \psi_\alpha \rangle_k$, which is positive definite (i.e., $U_\alpha(k) \geq 0$). Then we note that $2|C_{\alpha\beta}(k)| \leq U_\alpha(k) + U_\beta(k)$. We can use this inequality to show that if the matrix $L_{\alpha\beta}(k)$ satisfies the diagonal dominance condition

$$L_{\alpha\beta}(k) \geq 0, \text{ for } \alpha \neq \beta, \tag{3.15}$$

$$2L_{\alpha\alpha}(k) + \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} (L_{\alpha\beta}(k) + L_{\beta\alpha}(k)) \leq 0, \tag{3.16}$$

then the energy spectrum $E(k)$ is positive definite. We give the proof in Appendix A. Both the two-layer quasi-geostrophic model and the multi-layer quasi-geostrophic model satisfy this diagonal dominance condition. As for the layer-by-layer potential enstrophy spectra $G_\alpha(k)$, it is immediately obvious that they are unconditionally positive definite, regardless of the form of the matrix $L_{\alpha\beta}(k)$, since by definition $G_\alpha(k) = \langle q_\alpha, q_\alpha \rangle_k$.

For the argument of the present paper we need the dissipation rates $D_E(k)$ and $D_G(k)$ for the energy and total potential enstrophy expressed in terms of the streamfunction spectrum $C_{\alpha\beta}(k)$. In appendix B we show that the energy dissipation rate spectrum $D_E(k)$ and the layer-by-layer potential enstrophy dissipation rate spectra $D_{G_\alpha}(k)$ are given by

$$D_E(k) = 2\sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k), \tag{3.17}$$

$$D_{G_\alpha}(k) = -2\sum_{\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k). \tag{3.18}$$

Note that in order for the dissipation terms to be truly dissipative, the dissipation spectra $D_E(k)$ and $D_G(k)$ need to be both positive definite. From the general form of the above equations this is not readily obvious. However, for simpler configurations of the dissipation operators, the above expressions for $D_E(k)$ and $D_G(k)$ simplify considerably, thereby making it possible to establish positive definiteness.

We restrict our attention to cases where the dissipation operators at every layer involve only the fields of the corresponding layer, with no explicit interlayer terms. This can be arranged in terms of a linear operator $\mathscr{D}_\alpha$ applied to either the streamfunction ψ_α or the potential vorticity q_α. If $D_\alpha(k)$ is the spectrum of the positive-definite operator $\mathscr{D}_\alpha$, then for the case of a dissipation term $d_\alpha = \mathscr{D}_\alpha \psi_\alpha$, we have $D_{\alpha\beta}(k) = \delta_{\alpha\beta} D_\beta(k)$. We designate this case as *streamfunction-dissipation*. The $D_E(k)$ and $D_{G_\alpha}(k)$ simplify as:

$$D_E(k) = 2\sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k) = 2\sum_{\alpha\beta} \delta_{\alpha\beta} D_\beta(k) C_{\alpha\beta}(k) \tag{3.19}$$

$$= 2\sum_\alpha D_\alpha(k) C_{\alpha\alpha}(k) = 2\sum_\alpha D_\alpha(k) U_\alpha(k), \tag{3.20}$$

$$D_{G_\alpha}(k) = -2\sum_{\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k) = -2\sum_{\beta\gamma} L_{\alpha\beta}(k) \delta_{\alpha\gamma} D_\gamma(k) C_{\beta\gamma}(k). \tag{3.21}$$

$$= -2\sum_\beta L_{\alpha\beta}(k)D_\beta(k)C_{\alpha\beta}(k). \tag{3.22}$$

Note that for $D_\alpha(k) > 0$, it follows that $D_E(k) \geq 0$, but it is not obvious that the same result extends to $D_{G_\alpha}(k)$. However, if we further assume that the same operator is used for all layers, i.e. $D_\alpha(k) = D(k)$, then we have the more specialized case of *symmetric streamfunction-dissipation*, and the dissipation rate spectra $D_E(k)$ and $D_G(k)$ can be simplified further to give:

$$D_E(k) = 2\sum_\alpha D_\alpha(k)U_\alpha(k) = 2D(k)\sum_\alpha U_\alpha(k) = 2D(k)U(k), \tag{3.23}$$

$$D_G(k) = \sum_\alpha D_{G_\alpha}(k) = -2\sum_{\alpha\beta} L_{\alpha\beta}(k)D_\beta(k)C_{\alpha\beta}(k) \tag{3.24}$$

$$= 2D(k)\left[-\sum_{\alpha\beta} L_{\alpha\beta}(k)C_{\alpha\beta}(k)\right] = 2D(k)E(k). \tag{3.25}$$

Now, $D(k) > 0$ implies both $D_E(k) \geq 0$ and $D_G(k) \geq 0$.

Another possible arrangement is *potential vorticity dissipation* in which the positive-definite operator $\mathscr{D}_\alpha$ is applied to the potential vorticity q_α, thereby yielding a dissipation term of the form

$$d_\alpha = -\mathscr{D}_\alpha q_\alpha = -\sum_\beta \mathscr{D}_\alpha \mathscr{L}_{\alpha\beta}\psi_\beta = \sum_\beta \mathscr{D}_{\alpha\beta}\psi_\beta. \tag{3.26}$$

It is easy to see that in this case, $D_{\alpha\beta}(k) = -D_\alpha(k)L_{\alpha\beta}(k)$, and consequently, the dissipation rate spectra $D_E(k)$ and $D_G(k)$ simplify as follows:

$$D_E(k) = 2\sum_{\alpha\beta} D_{\alpha\beta}(k)C_{\alpha\beta}(k) = -2\sum_{\alpha\beta} D_\alpha(k)L_{\alpha\beta}(k)C_{\alpha\beta}(k), \tag{3.27}$$

$$D_{G_\alpha}(k) = -2\sum_{\beta\gamma} L_{\alpha\beta}(k)D_{\alpha\gamma}(k)C_{\beta\gamma}(k) = -2\sum_{\beta\gamma} L_{\alpha\beta}(k)[-D_\alpha(k)L_{\alpha\gamma}(k)]C_{\beta\gamma}(k) \tag{3.28}$$

$$= 2D_\alpha(k)\left[\sum_{\beta\gamma} L_{\alpha\beta}(k)L_{\alpha\gamma}(k)C_{\beta\gamma}(k)\right] = 2D_\alpha(k)G_\alpha(k). \tag{3.29}$$

Now, for $D_\alpha(k) > 0$, it follows that $D_{G_\alpha}(k) \geq 0$, whereas it is now the sign of $D_E(k)$ that is uncertain. Similarly to the previous case, if we assume *symmetric potential vorticity dissipation*, where the same dissipation operator is used on all layers, we then set $D_\alpha(k) = D(k)$, and our expression for the dissipation rate spectra $D_E(k)$ and $D_G(k)$ simplify to

$$D_E(k) = -2\sum_{\alpha\beta} D_\alpha(k)L_{\alpha\beta}(k)C_{\alpha\beta}(k) \tag{3.30}$$

$$= 2D(k)\left[-\sum_{\alpha\beta} L_{\alpha\beta}(k)C_{\alpha\beta}(k)\right] = 2D(k)E(k), \tag{3.31}$$

$$D_G(k) = 2\sum_\alpha D_\alpha(k)G_\alpha(k) = 2D(k)\sum_\alpha G_\alpha(k) = 2D(k)G(k). \tag{3.32}$$

Now, for $D(k) > 0$, it follows again that both $D_E(k) \geq 0$ and $D_G(k) \geq 0$. Note that the negative sign in $-\mathscr{D}_\alpha q_\alpha$ was necessary to ensure that the dissipation term $d_\alpha = -\mathscr{D}_\alpha q_\alpha$ remains dissipative when the operator $\mathscr{D}_\alpha$ is positive-definite.

4. Flux Inequality for the two-layer model

We now turn to the main issue of identifying sufficient conditions for satisfying the flux inequality $k^2\Pi_E(k) - \Pi_G(k) \leq 0$ for quasi-geostrophic models. Presently, we restrict our interest to the two-layer quasi-geostrophic model under symmetric or asymmetric streamfunction-dissipation or potential vorticity-dissipation. The multilayer case and other configurations for the dissipation terms will be considered in future publications. We begin with a brief review of the two-dimensional Navier-Stokes case and the formal setup of the argument in section 4.1. Our calculations for the case of streamfunction dissipation are given in section 4.2, and for the case of potential enstrophy dissipation in section 4.3. Since the details of the calculations are somewhat technical, a summary discussion of the main results is given in section 5.

4.1. *Preliminaries*

Let us recall that the energy flux spectrum $\Pi_E(k)$ is defined as the amount of energy transferred from the $(0, k)$ interval to the $(k, +\infty)$ interval per unit time and per unit volume. Likewise, the potential enstrophy flux spectrum $\Pi_G(k)$ is the amount of potential enstrophy transferred from the $(0, k)$ interval to the $(k, +\infty)$ interval, again per unit time and volume. Assuming a forced-dissipative configuration at steady state and that the wavenumber k is not in the forcing range, the energy and potential enstrophy transferred into the $(k, +\infty)$ interval eventually are dissipated somewhere in that interval. It follows that we may write the flux spectra $\Pi_E(k)$ and $\Pi_G(k)$ as integrals of the energy and potential enstrophy dissipation rate spectra $D_E(k)$ and $D_G(k)$:

$$\Pi_E(k) = \int_k^{+\infty} D_E(q)\mathrm{d}q, \tag{4.1}$$

$$\Pi_G(k) = \int_k^{+\infty} D_G(q)\mathrm{d}q, \tag{4.2}$$

which implies that

$$k^2\Pi_E(k) - \Pi_G(k) = \int_k^{+\infty} [k^2 D_E(q) - D_G(q)]\mathrm{d}q = \int_k^{+\infty} \Delta(k, q)\mathrm{d}q. \tag{4.3}$$

We see that a sufficient condition for establishing the flux inequality is to show that $\Delta(k, q) \leq 0$ for all wavenumbers $k < q$. It is also easy to see that $\Delta(k, q) > 0$ for all wavenumbers $k_t < k < q$ is sufficient for establishing the violation of the flux inequality for all wavenumbers $k > k_t$.

For the case of two-dimensional Navier-Stokes turbulence, the dissipation rate spectra $D_E(k)$ and $D_G(k)$ are related via $D_G(k) = k^2 D_E(k)$. This immediately gives $\Delta(k, q) = k^2 D_E(q) - D_G(q) = (k^2 - q^2)D_E(q) \leq 0$ for all wavenumbers $k < q$ (since $D_E(k) \geq 0$), which in turn gives the flux inequality $k^2\Pi_E(k) - \Pi_G(k) \leq 0$. The physical interpretation of this inequality is that when we stretch the separation of scales in the downscale range, the energy dissipation rate at small-scales vanishes rapidly. As a result, most of the injected energy cannot cascade downscale although, as noted by Gkioulekas & Tung (2005*a*,*b*), a small amount of energy is able to do so. As we have seen in the previous section, for the case of quasi-geostrophic models, the energy and potential enstrophy dissipation rate spectra no longer have a direct and simple relation with each other, so the validity of the flux inequality needs to be carefully re-examined.

We set up our analysis of the two-layer quasi-geostrophic model by writing the layer

interaction matrix $L_{\alpha\beta}(k)$ as

$$L_{\alpha\beta}(k) = -\begin{bmatrix} a(k) & b(k) \\ b(k) & a(k) \end{bmatrix}, \tag{4.4}$$

with $a(k)$ and $b(k)$ given by $a(k) = k^2 + k_R^2/2$ and $b(k) = -k_R^2$. As will become apparent from the argument below, retaining generality and calculating $\Delta(k, q)$ in terms of $a(k)$ and $b(k)$ seems to simplify the details of our calculation. From Eq. (3.11), we write the energy spectrum $E(k)$ in terms of the streamfunction spectra $U_1(k)$, $U_2(k)$, and $C_{12}(k)$:

$$E(k) = -\sum_{\alpha\beta} L_{\alpha\beta}(k) C_{\alpha\beta}(k) \tag{4.5}$$

$$= a(k)C_{11}(k) + b(k)C_{12}(k) + b(k)C_{21}(k) + a(k)C_{22}(k) \tag{4.6}$$

$$= a(k)[U_1(k) + U_2(k)] + 2b(k)C_{12}(k) \tag{4.7}$$

$$= [a(k) + b(k)]U(k) + b(k)[2C_{12}(k) - U(k)]. \tag{4.8}$$

Note that we reorganize the expression in terms of $2C_{12}(k) - U(k)$ because it is known that $2C_{12}(k) - U(k) \leq 0$ for all wavenumbers k. Likewise, we also know that $U(k) \geq 0$. Working with spectra that are known to be positive definite or negative definite, makes it easier to determine the sign of $\Delta(k, q)$ in the calculations below. Similarly, from Eq. (3.14), the layer-by-layer potential enstrophy spectrum $G_\alpha(k)$ reads

$$G_\alpha(k) = \sum_{\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) C_{\beta\gamma}(k) \tag{4.9}$$

$$= L_{\alpha 1}(k)L_{\alpha 1}(k)C_{11}(k) + 2L_{\alpha 1}(k)L_{\alpha 2}(k)C_{12}(k) + L_{\alpha 2}(k)L_{\alpha 2}(k)C_{22}(k), \tag{4.10}$$

and it follows that for the top and bottom layers

$$G_1(k) = a^2(k)U_1(k) + 2a(k)b(k)C_{12}(k) + b^2(k)U_2(k), \tag{4.11}$$

$$G_2(k) = b^2(k)U_1(k) + 2a(k)b(k)C_{12}(k) + a^2(k)U_2(k). \tag{4.12}$$

The total potential enstrophy spectrum $G(k) = G_1(k) + G_2(k)$ then reads:

$$G(k) = [a^2(k) + b^2(k)]U(k) + 4a(k)b(k)C_{12}(k). \tag{4.13}$$

We use the above equations to evaluate the dissipation rate spectra for the cases of streamfunction dissipation and potential vorticity streamfunction.

4.2. *Case 1: Streamfunction dissipation*

We begin with considering the case of streamfunction dissipation in which the top-layer has a dissipation operator with spectrum $D_1(k) = D(k)$ and the bottom layer has dissipation operator with spectrum $D_2(k) = D(k) + d(k)$. For $d(k) = 0$ we have the case of symmetric dissipation in which both layers have the same dissipation operators. For $d(k) > 0$, we have the more general case of asymmetric dissipation. From Eq. (3.20), the energy dissipation rate spectrum $D_E(k)$ is given by

$$D_E(k) = 2\sum_\alpha D_\alpha(k) U_\alpha(k) \tag{4.14}$$

$$= 2\{D(k)U_1(k) + [D(k) + d(k)]U_2(k)\}. \tag{4.15}$$

This equation can be rewritten in the form

$$D_E(k) = A_E^{(1)}(k)U_1(k) + A_E^{(2)}(k)U_2(k) + A_E^{(3)}(k)[2C_{12}(k) - U(k)], \tag{4.16}$$

with $A_E^{(1)}(k)$ and $A_E^{(2)}(k)$ given by

$$A_E^{(1)}(k) = 2D(k) \text{ and } A_E^{(2)}(k) = 2[D(k) + d(k)] \text{ and } A_E^{(3)}(k) = 0. \tag{4.17}$$

It can also be rewritten in the form

$$D_E(k) = B_E^{(1)}(k)D(k) + B_E^{(2)}(k)d(k), \tag{4.18}$$

with $B_E^{(1)}(k)$ and $B_E^{(2)}(k)$ given by

$$B_E^{(1)}(k) = 2U_1(k) + 2U_2(k) = 2U(k) \text{ and } B_E^{(2)}(k) = 2U_2(k). \tag{4.19}$$

Similarly, from Eq. (3.22), the potential enstrophy dissipation rate spectrum $D_G(k)$ is given by

$$\begin{aligned}
D_G(k) &= -2\sum_{\alpha\beta} L_{\alpha\beta}(k)D_\beta(k)C_{\alpha\beta}(k) &(4.20)\\
&= 2D(k)\left[-\sum_{\alpha\beta} L_{\alpha\beta}(k)C_{\alpha\beta}(k)\right] + 2d(k)\left[-\sum_{\beta} L_{2\beta}(k)C_{2\beta}(k)\right] &(4.21)\\
&= 2D(k)E(k) + 2d(k)[b(k)C_{21}(k) + a(k)U_2(k)] &(4.22)\\
&= 2D(k)E(k) + d(k)b(k)[2C_{12}(k) - U(k)] + d(k)b(k)U(k)\\
&\quad + 2d(k)a(k)U_2(k) &(4.23)\\
&= 2D(k)E(k) + d(k)b(k)[2C_{12}(k) - U(k)] + d(k)b(k)U_1(k)\\
&\quad + d(k)[2a(k) + b(k)]U_2(k). &(4.24)
\end{aligned}$$

Substituting the energy spectrum $E(k)$ from Eq. (4.8), it immediately follows that $D_G(k)$ reads

$$D_G(k) = A_G^{(1)}(k)U_1(k) + A_G^{(2)}(k)U_2(k) + A_G^{(3)}(k)[2C_{12}(k) - U(k)], \tag{4.25}$$

with $A_G^{(1)}(k)$, $A_G^{(2)}(k)$, and $A_G^{(3)}(k)$ given by

$$\begin{aligned}
A_G^{(1)}(k) &= 2D(k)[a(k) + b(k)] + d(k)b(k), &(4.26)\\
A_G^{(2)}(k) &= 2D(k)[a(k) + b(k)] + d(k)[2a(k) + b(k)], &(4.27)\\
A_G^{(3)}(k) &= 2D(k)b(k) + d(k)b(k) = [2D(k) + d(k)]b(k). &(4.28)
\end{aligned}$$

The potential enstrophy dissipation rate spectrum can also be reorganized in terms of $D(k)$ and $d(k)$ and be rewritten as

$$D_G(k) = B_G^{(1)}(k)D(k) + B_G^{(2)}(k)d(k), \tag{4.29}$$

with $B_G^{(1)}(k)$ and $B_G^{(2)}(k)$ given by

$$\begin{aligned}
B_G^{(1)}(k) &= 2[a(k) + b(k)]U(k) + 2b(k)[2C_{12}(k) - U(k)], &(4.30)\\
B_G^{(2)}(k) &= 2[b(k)C_{12}(k) + a(k)U_2(k)]. &(4.31)
\end{aligned}$$

From the above equations we may write the terms of $\Delta(k, q)$ in terms of the streamfunction spectra $U_1(k)$, $U_2(k)$, and $C_{12}(k)$, or in terms of the dissipation operator spectra $D(k)$ and $d(k)$. In the former case, we find that $\Delta(k, q)$ reads:

$$\begin{aligned}
\Delta(k, q) &= k^2 D_E(q) - D_G(q) &(4.32)\\
&= A_1(k, q)U_1(q) + A_2(k, q)U_2(q) + A_3(k, q)[2C_{12}(q) - U(q)], &(4.33)
\end{aligned}$$

with $A_1(k,q)$, $A_2(k,q)$, $A_3(k,q)$ given by

$$A_1(k,q) = k^2 A_E^{(1)}(q) - A_G^{(1)}(q) \tag{4.34}$$
$$= k^2[2D(q)] - 2D(q)[a(q)+b(q)] - d(q)b(q) \tag{4.35}$$
$$= 2D(q)[k^2 - a(q) - b(q)] - d(q)b(q), \tag{4.36}$$
$$A_2(k,q) = k^2 A_E^{(2)}(q) - A_G^{(2)}(q) \tag{4.37}$$
$$= k^2 2[D(q)+d(q)] - 2D(q)[a(q)+b(q)] - d(q)[2a(q)+b(q)] \tag{4.38}$$
$$= 2D(q)[k^2 - a(q) - b(q)] + d(q)[2k^2 - 2a(q) - b(q)] \tag{4.39}$$
$$= 2[D(q)+d(q)][k^2 - a(q) - b(q)] + d(q)b(q), \tag{4.40}$$
$$A_3(k,q) = k^2 A_E^{(3)}(q) - A_G^{(3)}(q) = -[2D(q)+d(q)]b(q). \tag{4.41}$$

Reorganizing the terms in the above expression in terms of $D(k)$ and $d(k)$, we can also rewrite $\Delta(k,q)$ as

$$\Delta(k,q) = B_1(k,q)D(q) + B_2(k,q)d(q), \tag{4.42}$$

with $B_1(k,q)$, $B_2(k,q)$ given by

$$B_1(k,q) = k^2 B_E^{(1)}(q) - B_G^{(1)}(q) \tag{4.43}$$
$$= k^2[2U(q)] - 2[a(q)+b(q)]U(q) - 2b(q)[2C_{12}(q) - U(q)] \tag{4.44}$$
$$= 2[k^2 - a(q) - b(q)]U(q) - 2b(q)[2C_{12}(q) - U(q)], \tag{4.45}$$
$$B_2(k,q) = k^2 B_E^{(2)}(q) - B_G^{(2)}(q) \tag{4.46}$$
$$= k^2[2U_2(q)] - 2[b(q)C_{12}(q) + a(q)U_2(q)] \tag{4.47}$$
$$= 2[k^2 - a(q)]U_2(q) - 2b(q)C_{12}(q). \tag{4.48}$$

Using the above expressions for $\Delta(k,q)$, we will now prove the following statements: Proposition 1 establishes the flux inequality for the case of symmetric streamfunction dissipation. Proposition 2 gives a sufficient condition in terms of the dissipation operator spectra $D(k)$ and $d(k)$ for satisfying the flux equality for the case of asymmetric streamfunction dissipation. Proposition 3 provides with an alternate sufficient condition for the asymmetric case formulated in terms of the streamfunction spectra $C_{12}(q)$ and $U_2(q)$.

PROPOSITION 1. *Assume streamfunction dissipation with $d(k) = 0$ and $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$. Then $\Delta(k,q) \leq 0$.*

Proof. Under the assumption $d(k) = 0$ (symmetric dissipation), $\Delta(k,q)$ reads:

$$\Delta(k,q) = A_1(k,q)U_1(q) + A_2(k,q)U_2(q) + A_3(k,q)[2C_{12}(q) - U(q)], \tag{4.49}$$

with $A_1(k,q)$, $A_2(k,q)$, and $A_3(k,q)$ given by

$$A_1(k,q) = 2D(q)[k^2 - a(q) - b(q)], \tag{4.50}$$
$$A_2(k,q) = 2D(q)[k^2 - a(q) - b(q)], \tag{4.51}$$
$$A_3(k,q) = -2D(q)b(q). \tag{4.52}$$

Now we note that $D(q) > 0$, and by hypothesis, $k^2 - a(q) - b(q) < 0$, and therefore we immediately obtain that $A_1(k,q) < 0$ and $A_2(k,q) < 0$. We also know, by hypothesis, that $b(q) < 0$, and therefore we also have $A_3(k,q) > 0$. Finally, we know that $U_1(q) \geq 0$ and $U_2(q) \geq 0$ and $2C_{12}(q) - U(q) \leq 0$. It follows that all three terms in Eq. (4.49) are negative, and therefore $\Delta(k,q) \geq 0$. □

Proposition 2. *Assume streamfunction dissipation with $d(k) > 0$ and $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$ and*

$$\frac{d(q)}{2D(q)} \leq \frac{k^2 - a(q) - b(q)}{b(q)}. \tag{4.53}$$

Then it follows that $\Delta(k,q) \leq 0$.

Proof. We now write $\Delta(k,q)$ as

$$\Delta(k,q) = A_1(k,q)U_1(q) + A_2(k,q)U_2(q) + A_3(k,q)[2C_{12}(q) - U(q)], \tag{4.54}$$

with $A_1(k,q)$, $A_2(k,q)$, and $A_3(k,q)$ given by

$$A_1(k,q) = 2D(q)[k^2 - a(q) - b(q)] - d(q)b(q), \tag{4.55}$$

$$A_2(k,q) = 2[D(q) + d(q)][k^2 - a(q) - b(q)] + d(q)b(q), \tag{4.56}$$

$$A_3(k,q) = -[2D(q) + d(q)]b(q). \tag{4.57}$$

From the assumptions $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$, given by hypothesis, and the fact that $D(q) > 0$ and $d(q) > 0$, we see that $A_2(k,q) < 0$ and $A_3(k,q) > 0$. Since $U_2(q) \geq 0$ and $2C_{12}(q) - U(q) \leq 0$, it follows that the second and third terms of Eq. (4.54) are negative, and therefore $\Delta(k,q) \leq A_1(k,q)U_1(q)$. We also note that $U_1(q) \geq 0$ and the hypothesis given by Eq. (4.53) implies that $A_1(k,q) \leq 0$. It follows that $\Delta(k,q) \leq 0$. □

Proposition 3. *Assume streamfunction dissipation with $d(k) > 0$ and $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$ and $C_{12}(q) \leq U_2(q)$. Then, it follows that $\Delta(k,q) \leq 0$.*

Proof. We write $\Delta(k,q)$ as

$$\Delta(k,q) = B_1(k,q)D(q) + B_2(k,q)d(q), \tag{4.58}$$

with $B_1(k,q)$ and $B_2(k,q)$ given by

$$B_1(k,q) = 2[k^2 - a(q) - b(q)]U(q) - 2b(q)[2C_{12}(q) - U(q)], \tag{4.59}$$

$$B_2(k,q) = 2[k^2 - a(q)]U_2(q) - 2b(q)C_{12}(q). \tag{4.60}$$

From the hypotheses $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$, and also because $U(q) \geq 0$ and $2C_{12}(q) - U(q) \leq 0$, we immediately find that $B_1(k,q) \leq 0$. From the hypothesis $C_{12}(q) \leq U_2(q)$, we can also show that

$$B_2(k,q) = 2[k^2 - a(q)]U_2(q) - 2b(q)C_{12}(q) \tag{4.61}$$

$$\leq 2[k^2 - a(q)]U_2(q) - 2b(q)U_2(q) \tag{4.62}$$

$$= 2[k^2 - a(q) - b(q)]U_2(q) \leq 0. \tag{4.63}$$

Since $d(q) > 0$ and $D(q) > 0$, it follows that $\Delta(k,q) \leq 0$. □

With respect to the hypotheses in the above propositions, we note that for the case of the two-layer quasi-geostrophic model, with $a(q) = q^2 + k_R^2/2$ and $b(q) = -k_R^2/2$, the assumptions $k^2 - a(q) - b(q) = k^2 - q^2 < 0$ and $b(q) < 0$ hold mathematically for all $k < q$. The assumption given by Eq. (4.53) in proposition 2 and the assumption $C_{12}(q) \leq U_2(q)$ in proposition 3 are the physically substantive assumptions.

4.3. *Case 2: Potential vorticity dissipation*

The case of potential vorticity dissipation is mainly an interesting mathematical curiosity. We are assuming that every layer is being dissipated by a dissipation operator with

positive definite spectrum $D_\alpha(k)$, applied on the potential vorticity field q_α of the corresponding layer, instead of the streamfunction ψ_α. The idea here is to determine whether this simple change in the configuration of the dissipation terms has any notable impact on the robustness of the flux inequality. As we have explained previously, in this case, the dissipation matrix $D_{\alpha\beta}(k)$ is given by $D_{\alpha\beta}(k) = -D_\alpha(k)L_{\alpha\beta}(k)$, and the energy dissipation rate spectrum $D_E(k)$ and the potential enstrophy dissipation rate spectrum $D_G(k)$ are given by Eq. (3.27) and Eq. (3.29) respectively.

Now, let us restrict ourselves again to the case of a general two-layer model with $D_1(k) = D(k)$ and $D_2(k) = D(k) + d(k)$. From Eq. (3.27), we calculate the energy dissipation rate spectrum $D_E(k)$ which reads

$$D_E(k) = -2\sum_{\alpha\beta} D_\alpha(k)L_{\alpha\beta}(k)C_{\alpha\beta}(k) \tag{4.64}$$

$$= 2D(k)\left[-\sum_{\alpha\beta} L_{\alpha\beta}(k)C_{\alpha\beta}(k)\right] + 2d(k)[-L_{21}(k)C_{21}(k) - L_{22}(k)C_{22}(k)] \tag{4.65}$$

$$= 2D(k)E(k) + 2d(k)[b(k)C_{12}(k) + a(k)U_2(k)]. \tag{4.66}$$

Similarly, from Eq. (3.29), we calculate the potential enstrophy dissipation rate spectrum $D_G(k)$, which is given by

$$D_G(k) = 2\sum_{\alpha} D_\alpha(k)G_\alpha(k) = 2D(k)G_1(k) + 2[D(k) + d(k)]G_2(k) \tag{4.67}$$

$$= 2D(k)[G_1(k) + G_2(k)] + 2d(k)G_2(k). \tag{4.68}$$

Both spectra may therefore be rewritten according to the form

$$D_E(k) = D(k)B_E^{(1)}(k) + d(k)B_E^{(2)}(k), \tag{4.69}$$

$$D_G(k) = D(k)B_G^{(1)}(k) + d(k)B_G^{(2)}(k), \tag{4.70}$$

with $B_E^{(1)}(k)$, $B_E^{(2)}(k)$, $B_G^{(1)}(k)$, and $B_G^{(2)}(k)$ given by:

$$B_E^{(1)}(k) = 2E(k) = 2[a(k)U(k) + 2b(k)C_{12}(k)] \tag{4.71}$$

$$= 2[a(k) + b(k)]U(k) + 2b(k)[2C_{12}(k) - U(k)], \tag{4.72}$$

$$B_E^{(2)}(k) = 2[b(k)C_{12}(k) + a(k)U_2(k)] \tag{4.73}$$

$$= b(k)[2C_{12}(k) - U(k)] + b(k)U(k) + 2a(k)U_2(k) \tag{4.74}$$

$$= b(k)U_1(k) + [2a(k) + b(k)]U_2(k) + b(k)[2C_{12}(k) - U(k)], \tag{4.75}$$

and

$$B_G^{(1)}(k) = 2[G_1(k) + G_2(k)] \tag{4.76}$$

$$= 2[a^2(k) + b^2(k)]U(k) + 8a(k)b(k)C_{12}(k) \tag{4.77}$$

$$= 2[a^2(k) + b^2(k)]U(k) + 4a(k)b(k)[2C_{12}(k) - U(k)] + 4a(k)b(k)U(k) \tag{4.78}$$

$$= 2[a(k) + b(k)]^2U(k) + 4a(k)b(k)[2C_{12}(k) - U(k)], \tag{4.79}$$

$$B_G^{(2)}(k) = 2G_2(k) = 2b^2(k)U_1(k) + 4a(k)b(k)C_{12}(k) + 2a^2(k)U_2(k) \tag{4.80}$$

$$\begin{aligned}&= 2a(k)b(k)[2C_{12}(k) - U(k)] + 2a(k)b(k)U(k) + 2b^2(k)U_1(k)\\ &\quad + 2a^2(k)U_2(k)\end{aligned} \tag{4.81}$$

$$= 2[a(k)b(k) + b^2(k)]U_1(k) + 2[a(k)b(k) + a^2(k)]U_2(k)$$

$$+ 2a(k)b(k)[2C_{12}(k) - U(k)] \tag{4.82}$$

$$= 2b(k)[a(k)+b(k)]U_1(k) + 2a(k)[a(k)+b(k)]U_2(k) + 2a(k)b(k)[2C_{12}(k) - U(k)]. \tag{4.83}$$

Equivalently, the terms of $D_E(k)$ and $D_G(k)$ can be reorganized in terms of the streamfunction spectra $U_1(q)$, $U_2(q)$, and $2C_{12}(q) - U(q)$, and rewritten as

$$D_E(k) = A_E^{(1)}(k)U_1(k) + A_E^{(2)}(k)U_2(k) + A_E^{(3)}(k)[2C_{12}(k) - U(k)], \tag{4.84}$$

$$D_G(k) = A_G^{(1)}(k)U_1(k) + A_G^{(2)}(k)U_2(k) + A_G^{(3)}(k)[2C_{12}(k) - U(k)], \tag{4.85}$$

with $A_E^{(1)}(k)$, $A_E^{(2)}(k)$, $A_E^{(3)}(k)$, $A_G^{(1)}(k)$, $A_G^{(2)}(k)$, and $A_G^{(3)}(k)$, given by

$$A_E^{(1)}(k) = 2D(k)[a(k)+b(k)] + d(k)b(k), \tag{4.86}$$

$$A_E^{(2)}(k) = 2D(k)[a(k)+b(k)] + d(k)[2a(k)+b(k)], \tag{4.87}$$

$$A_E^{(3)}(k) = 2D(k)b(k) + d(k)b(k), \tag{4.88}$$

and

$$A_G^{(1)}(k) = 2D(k)[a(k)+b(k)]^2 + 2d(k)b(k)[a(k)+b(k)], \tag{4.89}$$

$$A_G^{(2)}(k) = 2D(k)[a(k)+b(k)]^2 + 2d(k)a(k)[a(k)+b(k)], \tag{4.90}$$

$$A_G^{(3)}(k) = 4D(k)a(k)b(k) + 2d(k)a(k)b(k). \tag{4.91}$$

From the above equations we now write the terms of $\Delta(k,q)$, collected in terms of the streamfunction spectra, as:

$$\Delta(k,q) = k^2 D_E(q) - D_G(q) \tag{4.92}$$

$$= A_1(k,q)U_1(q) + A_2(k,q)U_2(q) + A_3(k,q)[2C_{12}(q) - U(q)], \tag{4.93}$$

with $A_1(k,q)$, $A_2(k,q)$, $A_3(k,q)$ given as

$$A_1(k,q) = k^2 A_E^{(1)}(q) - A_G^{(1)}(q) \tag{4.94}$$

$$= 2k^2 D(q)[a(q)+b(q)] + k^2 d(q)b(q) - 2D(q)[a(q)+b(q)]^2 - 2d(q)b(q)[a(q)+b(q)] \tag{4.95}$$

$$= 2D(q)[a(q)+b(q)][k^2 - a(q) - b(q)] + d(q)b(q)[k^2 - 2a(q) - 2b(q)], \tag{4.96}$$

$$A_2(k,q) = k^2 A_E^{(2)}(q) - A_G^{(2)}(q) \tag{4.97}$$

$$= 2k^2 D(q)[a(q)+b(q)] + k^2 d(q)[2a(q)+b(q)] - 2D(q)[a(q)+b(q)]^2 - 2d(q)a(q)[a(q)+b(q)] \tag{4.98}$$

$$= 2D(q)[a(q)+b(q)][k^2 - a(q) - b(q)] + d(q)\{2k^2 a(q) + k^2 b(q) - 2a(q)[a(q)+b(q)]\} \tag{4.99}$$

$$= 2D(q)[a(q)+b(q)][k^2 - a(q) - b(q)] + d(q)\{2a(q)[k^2 - a(q) - b(q)] + k^2 b(q)\} \tag{4.100}$$

$$= [k^2 - a(q) - b(q)]\{2D(q)[a(q)+b(q)] + 2a(q)d(q)\} + k^2 d(q)b(q), \tag{4.101}$$

$$A_3(k,q) = k^2 A_E^{(3)}(q) - A_G^{(3)}(q) \tag{4.102}$$

$$= 2k^2 D(q)b(q) + k^2 d(q)b(q) - 4D(q)a(q)b(q) - 2d(q)a(q)b(q) \tag{4.103}$$

$$= 2D(q)b(q)[k^2 - 2a(q)] + d(q)b(q)[k^2 - 2a(q)] \tag{4.104}$$

$$= b(q)[2D(q) + d(q)][k^2 - 2a(q)]. \tag{4.105}$$

Similarly to the previous case of the streamfunction dissipation, we now derive the following two propositions. Proposition 4 establishes the flux equality for the case of symmetric potential vorticity dissipation. Proposition 5 provides a sufficient condition for satisfying the flux inequality, for the case of asymmetric potential vorticity dissipation.

PROPOSITION 4. *Assume potential vorticity dissipation with $d(k) = 0$ for all $k > 0$ and $a(q) + b(q) > 0$ and $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$. Then, it follows that $\Delta(k, q) \leq 0$.*

Proof. Under the assumption that $d(k) = 0$ for all wavenumbers k, the general form of $\Delta(k, q)$ and the coefficients $A_1(k, q)$, $A_2(k, q)$, $A_3(k, q)$ simplify to:

$$\Delta(k, q) = A_1(k, q)U_1(q) + A_2(k, q)U_2(q) + A_3(k, q)[2C_{12}(q) - U(q)], \tag{4.106}$$

$$A_1(k, q) = 2D(q)[a(q) + b(q)][k^2 - a(q) - b(q)], \tag{4.107}$$

$$A_2(k, q) = 2D(q)[a(q) + b(q)][k^2 - a(q) - b(q)], \tag{4.108}$$

$$A_3(k, q) = 2D(q)b(q)[k^2 - 2a(q)]. \tag{4.109}$$

From the assumptions $a(q) + b(q) > 0$ and $k^2 - a(q) - b(q) < 0$, it immediately follows that $A_1(k, q) < 0$, and $A_2(k, q) < 0$. We also find that $a(q) > -b(q) > 0$. Using this result in conjugation with the previous two assumptions, we show that

$$k^2 - 2a(q) = [k^2 - a(q) - b(q)] - a(q) + b(q) \tag{4.110}$$

$$< [k^2 - a(q) - b(q)] + b(q) \tag{4.111}$$

$$< k^2 - a(q) - b(q) < 0. \tag{4.112}$$

It follows that $b(q)(k^2 - 2a(q)) > 0$, and therefore $A_3(k, q) > 0$. Since $U_1(q) \geq 0$ and $U_2(q) \geq 0$ and $2C_{12}(q) - U(q) \leq 0$, it follows that all 3 terms in Eq. (4.106) are negative and therefore $\Delta(k, q) \leq 0$. □

PROPOSITION 5. *Assume potential vorticity dissipation and assume that $a(q)+b(q) > 0$ and $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$. We also assume that $d(q)$ satisfies*

$$\frac{d(q)}{D(q)} \leq \frac{-[a(q) + b(q)][k^2 - a(q) - b(q)]}{b(q)[k^2 - 2a(q) - 2b(q)]}. \tag{4.113}$$

Then, it follows that $\Delta(k, q) \leq 0$.

Proof. Recall that the general form of $\Delta(k, q)$ reads

$$\Delta(k, q) = A_1(k, q)U_1(q) + A_2(k, q)U_2(q) + A_3(k, q)[2C_{12}(q) - U(q)], \tag{4.114}$$

with $A_1(k, q)$, $A_2(k, q)$, $A_3(k, q)$ given by

$$A_1(k, q) = 2D(q)[a(q) + b(q)][k^2 - a(q) - b(q)] + d(q)b(q)[k^2 - 2a(q) - 2b(q)], \tag{4.115}$$

$$A_2(k, q) = [k^2 - a(q) - b(q)]\{2D(q)[a(q) + b(q)] + 2a(q)d(q)\} + k^2 d(q)b(q), \tag{4.116}$$

$$A_3(k, q) = b(q)[2D(q) + d(q)][k^2 - 2a(q)]. \tag{4.117}$$

As in the previous proof, our assumptions also imply that $a(q) > 0$ and $k^2 - 2a(q) < 0$. These are sufficient for determining the sign of $A_2(k, q)$ and $A_3(k, q)$. For the case of $A_2(k, q)$, we note that it consists of two terms, both of which positive, as the product of a negative and a positive factor. It follows that $A_2(k, q) < 0$. Likewise, $A_3(k, q)$ is the product of two negative and one positive factors, and therefore $A_3(k, q) > 0$. Since

$U_2(q) \geq 0$ and $2C_{12}(q) - U(q) \leq 0$, it follows that $\Delta(k,q) \leq A_1(k,q)U_1(q)$. For the case $A_1(k,q)$, we see that it consists of two competing terms: the first negative and the second positive. However, from Eq. (4.113), it follows that $A_1(k,q) < 0$, consequently $\Delta(k,q) \leq 0$, as claimed. □

Comparing the above propositions with propositions 1, 2, and 3, corresponding to the case of streamfunction dissipation, we see that the additional assumption $a(q)+b(q) > 0$ is needed. For the case of the two-layer quasi-geostrophic model, we have $a(q) + b(q) = q^2 > 0$, so the assumption is satisfied. We also note that Eq. (4.53) and Eq. (4.113) are not identical, but they are nonetheless very similar. Finally, I was not able to find an appropriate counterpart to proposition 3. Whether there exists such a counterpart remains an open question.

5. Summary of main results

In the previous section we derived a series of propositions that provide us with sufficient conditions for establishing the inequality $\Delta(k,q) \leq 0$ for all wavenumbers $q > k$. These results are mathematically rigorous and require no physical assumptions. As explained in the beginning, this inequality in turn establishes the flux inequality $k^2\Pi_E(k) - \Pi_G(k) \leq 0$. In this last step we have to assume that the wavenumber k is not in the forcing range. In quasi-geostrophic turbulence models, the forcing spectrum is controlled by the baroclinicity instability. Consequently, implicit in the argument below is the assumption that baroclinic instability is negligible at large wavenumbers. This has been originally proposed by Salmon (1978, 1980) and has been corroborated by Welch & Tung (1998) and Tung & Orlando (2003). For investigative purposes, in a numerical simulation, we can assume control of the forcing spectrum by using antisymmetric random gaussian forcing, as explained by Gkioulekas (2012).

Our results are restricted to the two-layer model under two distinct dissipation configurations: streamfunction dissipation and potential vorticity dissipation. For both configurations, we have shown that as long as the same dissipation operator is applied on both layers, the flux inequality is satisfied for all wavenumbers not in the forcing range. This follows from propositions 1 and 4. For the case of asymmetric dissipation, in which the operators for the two layers are not identical, we need to distinguish between the case of streamfunction dissipation and potential vorticity dissipation.

Under asymmetric streamfunction dissipation, in which the dissipation operator of the top layer has spectrum $D_1(k) = D(k)$ and the dissipation operator of the bottom layer has spectrum $D_2(k) = D(k) + d(k)$, we have shown by proposition 2 that

$$\frac{d(q)}{2D(q)} \leq \frac{k^2 - a(q) - b(q)}{b(q)} \Longrightarrow \Delta(k,q) \leq 0. \tag{5.1}$$

For the case of the two-layer model, using $a(q) = q^2 + k_R^2/2$ and $b(q) = -k_R^2/2$, gives $k^2 - a(q) - b(q) = k^2 - q^2$ and consequently the above statement reduces to

$$\frac{d(q)}{D(q)} \leq \frac{4(q^2-k^2)}{k_R^2} \Longrightarrow \Delta(k,q) \leq 0. \tag{5.2}$$

For the typical case in which both layers are dissipated by the same hyperdiffusion operator, and the bottom layer is dissipated by an Ekman term, we have $D(q) = \nu q^{2p+2}$ and $d(q) = \nu_E q^2$ which gives the following statement

$$k_R^2 \nu_E \leq 4\nu q^{2p}(q^2 - k^2) \Longrightarrow \Delta(k,q) \leq 0. \tag{5.3}$$

For finite Ekman coefficient ν_E, it is easy to see that the inequality is violated when k and q are too close to each other, so in principle, $\Delta(k, q)$ could become positive for $k \sim q$ and then transition to negative for $k \ll q$. On the other hand, if the wavenumber k is deep within the inertial range, the contributions of $q \sim k$ to the integral in Eq. (4.3) are bound to be negligible since dissipation is not expected to be dominant in the inertial range. Furthermore, the inequality only ensures the negativity of an upper bound of $\Delta(k, q)$, namely the contribution of the $A_1(k, q)U_1(q)$ term, which is the only term that can be positive or negative. Even if the $A_1(k, q)U_1(q)$ term is positive, for small enough Ekman coefficient ν_E, it is reasonable to expect it to be overtaken by the other two negative terms, since for $\nu_E = 0$, the $A_1(k, q)U_1(q)$ term is also negative. Consequently, we believe that for small enough ν_E, the flux inequality will continue to hold.

This is as much as can be said without making any phenomenological assumptions. If we had a more detailed knowledge of the phenomenology of the streamfunction spectra $U_1(q)$, $U_2(q)$, and $C_{12}(q)$, it would be possible to calculate $\Delta(k, q)$ explicitly and perform a more precise investigation. In connection with this matter of the phenomenology of the streamfunction spectra, it is worth noting that we have also shown via proposition 3 that another sufficient condition for establishing the flux inequality at wavenumber k is to have $C_{12}(q) < U_2(q)$ for all wavenumbers $q > k$.

For the case of potential vorticity dissipation, from proposition 5 we have a similar result that

$$\frac{d(q)}{D(q)} \leq \frac{-[a(q) + b(q)][k^2 - a(q) - b(q)]}{b(q)[k^2 - 2a(q) - 2b(q)]} \Longrightarrow \Delta(k, q) \leq 0, \tag{5.4}$$

and setting $a(q)$ and $b(q)$ to $a(q) = q^2 + k_R^2/2$ and $b(q) = -k_R^2/2$, as required for the case of the two-layer model, gives $a(q) + b(q) = q^2$, $k^2 - a(q) - b(q) = k^2 - q^2$, and $k^2 - 2a(q) - 2b(q) = k^2 - 2q^2$, and the general result reduces to

$$\frac{d(q)}{D(q)} \leq \frac{4(q^2 - k^2)}{k_R^2} \frac{q^2}{2q^2 - k^2} \Longrightarrow \Delta(k, q) \leq 0. \tag{5.5}$$

A typical choice for the dissipation operator spectra $D(k)$ and $d(k)$ is to choose $D(q) = \nu q^{2p}$ and $d(q) = \nu_E$. This choice gives the statement

$$\nu_E k_R^2 (2q^2 - k^2) \leq 4\nu q^{2p+2}(q^2 - k^2) \Longrightarrow \Delta(k, q) \leq 0. \tag{5.6}$$

Similar considerations apply to this case, as in our discussion above of the case of streamfunction dissipation. However, the sufficient condition here gives a tighter inequality, which suggests that a violation of the flux inequality may be easier under potential vorticity dissipation.

It is interesting to note that if one uses differential hyperdiffusion, whereby the hyperdissipation term at the bottom layer has greater hyperviscosity coefficient than the hyperdissipation term at the top layer, then for $D(k) = \nu q^{2p+2}$ and $d(k) = \nu_E q^2 + \Delta\nu q^{2p+2}$, we have, for the case of streamfunction dissipation, the statement

$$k_R^2 \nu_E \leq q^{2p}[4\nu(q^2 - k^2) - k_R^2 \Delta\nu] \Longrightarrow \Delta(k, q) \leq 0, \tag{5.7}$$

and, for the case of potential vorticity dissipation, using $D(k) = \nu q^{2p}$ and $d(k) = \nu_E + \Delta\nu q^{2p}$, the statement

$$\nu_E k_R^2 (2q^2 - k^2) \leq q^{2p}[(q^2 - k^2)(4\nu q^2 - \Delta\nu k_R^2) - 2k_R^2 \Delta\nu q^2] \Longrightarrow \Delta(k, q) \leq 0. \tag{5.8}$$

Here, we have assumed that the hyperviscosity coefficient at the top layer is ν and at the bottom layer is $\nu + \Delta\nu$ with $\Delta\nu > 0$. We see that in both cases, the inequality constraint on ν_E becomes tighter. We may therefore speculate that with increasing $\Delta\nu$, it may

become easier to break the flux inequality in the two-layer quasi-geostrophic model. This possibility warrants further numerical investigation.

6. Conclusion and Discussion

The flux inequality and the possibility of its violation was discussed previously in a less systematic manner by Gkioulekas & Tung (2007*a*). In the present paper, we have provided a more systematic treatment of the energy and potential enstrophy dissipation rate spectra for the general case of an arbitrary multi-layer model. We gave a more careful derivation of the sufficient conditions for satisfying the flux inequality, for the case of streamfunction dissipation, and extended these results to the case of potential vorticity dissipation. The careful form of our argument makes it possible to consider the possibility of differential hyperdiffusion, and we now have an ideal launching point for a more thorough investigation of the flux inequality. Our results on the non-trivial dependence of the dissipation rate spectra of energy and potential enstrophy on the energy and potential enstrophy spectra via the streamfunction spectra, are also very relevant to the correct formulation of closure models for multi-layer quasi-geostrophic systems. We would like to stress again that these results are mathematically rigorous and the only underlying assumption is that forcing, which is driven by baroclinic instability, is confined to large scales. This assumption is not needed to establish propositions 1-5, but it needs to be introduced when taking the next step to establish the flux inequality from the conclusion of these propositions. No phenomenological assumptions about any spectrum are needed at any step of the argument.

One limitation of the current investigation is that we have disregarded the beta term, mainly to avoid the mathematical difficulties associated with the anisotropic nature of the term. This elimination can be tolerated, from a physical standpoint, as long as the beta term is active only in the forcing range and the baroclinic forcing at the same forcing range is powerful enough to overshadow the beta term. If barotropization of the two-layer quasi-geostrophic model is indeed catalyzed by the beta term, as suggested by Venaille, Vallis & Griffies (2012), then the effect of the beta term will probably be to strengthen the robustness of the flux inequality, with increasing values of the β coefficient. This tendency can be resisted by asymmetric dissipation, but it is not obvious whether the two tendencies can counterbalance each other for any value of the β coefficient, or which of the two tendencies gets to dominate for realistic choices of β. On the other hand, as long as the effect of the beta term remains limited to large scales (i.e. planetary and synoptic scales), it will not contribute to the integrals of Eq. (4.1) and Eq. (4.2) and the results reported in this paper will remain entirely unaffected.

As was explained previously in the introduction of the present paper, the Tung & Orlando (2003) simulation indicates that the downscale energy cascade can indeed be uncovered. However, in the absence of the theory developed in the present paper and previous papers (Gkioulekas 2012; Gkioulekas & Tung 2005*a*,*b*, 2006, 2007*a*,*b*), Tung & Orlando (2003) were not motivated to explore the layer-by-layer phenomenology. A numerical investigation of this phenomenology would go a long way towards building our understanding of the dynamics of the two-layer quasi-geostrophic model, as a stepping stone for further investigation of multi-layer quasi-geostrophic systems. Just as two-dimensional turbulence research turned out to be interesting, exciting, and full of surprises, in its own right, we believe that quasi-geostrophic models, such as the two-layer model, will prove to have an even richer set of surprises in store for us. The two-layer model is a very good platform for investigating the dynamics of coexisting cascades, and for that reason alone, further investigation is warranted.

The idea of a flux inequality was first brought up in email communication between Sergey Danilov with the author and Ka-Kit Tung, in the context of two-dimensional Navier-Stokes turbulence.

Appendix A. Proof that $E(k)$ is positive definite

In this appendix we prove the inequality $2|C_{\alpha\beta}(k)| \leq U_\alpha(k) + U_\beta(k)$, and we use it to show that if the matrix $L_{\alpha\beta}(k)$ satisfies the following conditions:

$$L_{\alpha\beta}(k) \geq 0, \ \text{for } \alpha \neq \beta, \tag{A 1}$$

$$2L_{\alpha\alpha}(k) + \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} (L_{\alpha\beta}(k) + L_{\beta\alpha}(k)) \leq 0, \tag{A 2}$$

then the energy spectrum $E(k)$ will be positive definite with $E(k) \geq 0$.

To establish the inequality, we first note that

$$\langle \psi_\alpha \pm \psi_\beta, \psi_\alpha \pm \psi_\beta \rangle_k = \langle \psi_\alpha, \psi_\alpha \rangle_k \pm 2 \langle \psi_\alpha, \psi_\beta \rangle_k + \langle \psi_\beta, \psi_\beta \rangle_k \tag{A 3}$$

$$= U_\alpha(k) + U_\beta(k) \pm 2C_{\alpha\beta}(k). \tag{A 4}$$

Since $\langle \psi_\alpha \pm \psi_\beta, \psi_\alpha \pm \psi_\beta \rangle_k$ is by definition positive definite, it follows that $U_\alpha(k)+U_\beta(k)-2C_{\alpha\beta}(k) \geq 0$ and $U_\alpha(k) + U_\beta(k) + 2C_{\alpha\beta}(k) \geq 0$, and therefore we have

$$2|C_{\alpha\beta}(k)| \leq U_\alpha(k) + U_\beta(k). \tag{A 5}$$

Now, to show that the energy spectrum $E(k)$ is positive definite, we begin by rewriting Eq. (3.11) as follows:

$$2E(k) = -2\sum_{\alpha\beta} L_{\alpha\beta}(k)C_{\alpha\beta}(k) = -2\sum_{\alpha} L_{\alpha\alpha}(k)U_\alpha(k) - 2\sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} L_{\alpha\beta}(k)C_{\alpha\beta}(k) \tag{A 6}$$

$$= -2\sum_{\alpha} L_{\alpha\alpha}(k)U_\alpha(k) - \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} L_{\alpha\beta}(k)[U_\alpha(k) + U_\beta(k)] \tag{A 7}$$

$$- \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} L_{\alpha\beta}(k)[2C_{\alpha\beta}(k) - U_\alpha(k) - U_\beta(k)] \tag{A 8}$$

$$= -2\sum_{\alpha} L_{\alpha\alpha}(k)U_\alpha(k) - \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} [L_{\alpha\beta}(k) + L_{\beta\alpha}(k)]U_\alpha(k) \tag{A 9}$$

$$- \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} L_{\alpha\beta}(k)[2C_{\alpha\beta}(k) - U_\alpha(k) - U_\beta(k)] \tag{A 10}$$

$$= -\sum_{\alpha}\{2L_{\alpha\alpha}(k) + \sum_{\substack{\beta \\ \alpha\neq\beta}} [L_{\alpha\beta}(k) + L_{\beta\alpha}(k)]\}U_\alpha(k) \tag{A 11}$$

$$- \sum_{\substack{\alpha\beta \\ \alpha\neq\beta}} L_{\alpha\beta}(k)[2C_{\alpha\beta}(k) - U_\alpha(k) - U_\beta(k)]. \tag{A 12}$$

We note that $U_\alpha(k) \geq 0$, since $U_\alpha(k)$ is positive definite, and we have just shown that $2C_{\alpha\beta}(k) - U_\alpha(k) - U_\beta(k) \leq 0$. Combining these with the assumptions given by Eq. (A 1) and Eq. (A 2), we see that both terms in our expression for $2E(k)$ are positive and therefore $E(k) \geq 0$.

For the case of a two-layer model with general matrix $L_{\alpha\beta}(k)$ given by

$$L_{\alpha\beta}(k) = -\begin{bmatrix} a(k) & b(k) \\ b(k) & a(k) \end{bmatrix}, \tag{A 13}$$

the conditions given by Eq. (A 1) and Eq. (A 2) reduce to $b(k) \leq 0$ and $a(k) + b(k) \geq 0$. For a two-layer quasi-geostrophic model, we have $a(k) = k^2 + k_R^2/2$ and $b(k) = -k_R^2/2$ and both conditions are readily satisfied.

Appendix B. Derivation of dissipation rate spectra

In this appendix, we will show that the energy dissipation rate spectrum $D_E(k)$ and the layer-by-layer potential enstrophy dissipation rate spectra $D_{G_\alpha}(k)$ are given by

$$D_E(k) = 2\sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k), \tag{B 1}$$

$$D_{G_\alpha}(k) = -2\sum_{\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k). \tag{B 2}$$

The proof mirrors the argument used by Gkioulekas (2012) to derive the energy forcing spectrum and the potential enstrophy forcing spectrum for the same model. We begin by writing the governing equation for the streamfunction field ψ_α as

$$\frac{\partial \psi_\alpha}{\partial t} + \sum_\beta \mathscr{L}_{\alpha\beta}^{-1} J(\psi_\beta, q_\beta) = \sum_{\beta\gamma} \mathscr{L}_{\alpha\beta}^{-1} \mathscr{D}_{\beta\gamma}\psi_\gamma + \sum_\beta \mathscr{L}_{\alpha\beta}^{-1} f_\beta. \tag{B 3}$$

Differentiating the streamfunction spectrum $C_{\alpha\beta}(k)$ with respect to time gives

$$\frac{\partial C_{\alpha\beta}(k)}{\partial t} = \left\langle \frac{\partial \psi_\alpha}{\partial t}, \psi_\beta \right\rangle_k + \left\langle \psi_\alpha, \frac{\partial \psi_\beta}{\partial t} \right\rangle_k, \tag{B 4}$$

and we may write a governing equation for $C_{\alpha\beta}(k)$ in the form:

$$\frac{\partial C_{\alpha\beta}(k)}{\partial t} + \mathcal{T}_{\alpha\beta}(k) = -\mathcal{D}_{\alpha\beta}(k) + \mathcal{F}_{\alpha\beta}(k). \tag{B 5}$$

Here, $\mathcal{T}_{\alpha\beta}(k)$ is the contribution from the nonlinear Jacobian term, $\mathcal{D}_{\alpha\beta}(k)$ is the contribution from the dissipation term, and $\mathcal{F}_{\alpha\beta}(k)$ is the contribution from the forcing term. The dissipation term $\mathcal{D}_{\alpha\beta}(k)$ can now be obtained by replacing in Eq. (B 4) the streamfunction time-derivative $\partial\psi_\alpha/\partial t$ with the dissipation term $\sum_{\beta\gamma} \mathscr{L}_{\alpha\beta}^{-1}\mathscr{D}_{\beta\gamma}\psi_\gamma$. This gives

$$\mathcal{D}_{\alpha\beta}(k) = -\left\langle \sum_{\gamma\delta} \mathscr{L}_{\alpha\gamma}^{-1}\mathscr{D}_{\gamma\delta}\psi_\delta, \psi_\beta \right\rangle_k - \left\langle \psi_\alpha, \sum_{\gamma\delta} \mathscr{L}_{\beta\gamma}^{-1}\mathscr{D}_{\gamma\delta}\psi_\delta \right\rangle_k \tag{B 6}$$

$$= -\sum_{\gamma\delta} [L_{\alpha\gamma}^{-1}(k) D_{\gamma\delta}(k) C_{\beta\delta}(k) + L_{\beta\gamma}^{-1}(k) D_{\gamma\delta}(k) C_{\alpha\delta}(k)]. \tag{B 7}$$

We may now easily write the dissipation rate spectra $D_E(k)$ and $D_G(k)$ by applying on $\mathcal{D}_{\alpha\beta}(k)$ the linear operators indicated by Eq. (3.11) and Eq. (3.14). We therefore find that the energy dissipation rate energy spectrum $D_E(k)$ is given by

$$D_E(k) = -\sum_{\alpha\beta} L_{\alpha\beta}(k)\mathcal{D}_{\alpha\beta}(k) \tag{B 8}$$

$$= \sum_{\alpha\beta\gamma\delta} [L_{\alpha\beta}(k) L^{-1}_{\alpha\gamma}(k) D_{\gamma\delta}(k) C_{\beta\delta}(k) + L_{\alpha\beta}(k) L^{-1}_{\beta\gamma}(k) D_{\gamma\delta}(k) C_{\alpha\delta}(k)] \tag{B 9}$$

$$= \sum_{\beta\gamma\delta} \delta_{\beta\gamma} D_{\gamma\delta}(k) C_{\beta\delta}(k) + \sum_{\alpha\gamma\delta} \delta_{\alpha\gamma} D_{\gamma\delta}(k) C_{\alpha\delta}(k) \tag{B 10}$$

$$= \sum_{\beta\delta} D_{\beta\delta}(k) C_{\beta\delta}(k) + \sum_{\gamma\delta} D_{\gamma\delta}(k) C_{\gamma\delta}(k) = 2 \sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k). \tag{B 11}$$

The layer-by-layer potential enstrophy spectrum $D_{G_\alpha}(k)$ is likewise given by

$$D_{G_\alpha}(k) = \sum_{\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) \mathcal{D}_{\beta\gamma}(k) \tag{B 12}$$

$$= -\sum_{\beta\gamma\delta\varepsilon} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) [L^{-1}_{\beta\delta}(k) D_{\delta\varepsilon}(k) C_{\gamma\varepsilon}(k) + L^{-1}_{\gamma\delta}(k) D_{\delta\varepsilon}(k) C_{\beta\varepsilon}(k)] \tag{B 13}$$

$$= -\sum_{\gamma\delta\varepsilon} \delta_{\alpha\delta} L_{\alpha\gamma}(k) D_{\delta\varepsilon}(k) C_{\gamma\varepsilon}(k) - \sum_{\beta\delta\varepsilon} \delta_{\alpha\delta} L_{\alpha\beta}(k) D_{\delta\varepsilon}(k) C_{\beta\varepsilon}(k) \tag{B 14}$$

$$= -\sum_{\gamma\varepsilon} L_{\alpha\gamma}(k) D_{\alpha\varepsilon}(k) C_{\gamma\varepsilon}(k) - \sum_{\beta\varepsilon} L_{\alpha\beta}(k) D_{\alpha\varepsilon}(k) C_{\beta\varepsilon}(k) \tag{B 15}$$

$$= -2 \sum_{\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k). \tag{B 16}$$

The corresponding conservation laws read

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial t} = -D_E(k) + F_E(k), \tag{B 17}$$

$$\frac{\partial G(k)}{\partial t} + \frac{\partial \Pi_G(k)}{\partial t} = -D_G(k) + F_G(k). \tag{B 18}$$

We see that positive $D_E(k)$ and $D_G(k)$ correspond to the case where the dissipation terms are truly dissipative. This concludes the argument.

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